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COMMENT

Comment on Goltsev's stability analysis for the Parisi solution of the long-ranged spin glass model

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Abstract. Goltsev's recent stability analysis for Parisi's solution of the Sherrington–Kirkpatrick model is shown to be incorrect near T_c, and also in disagreement with the analysis by the present authors.

The question of stability is of crucial importance for the validity of the now widely accepted Parisi (1979, 1980) solution of the Sherrington–Kirkpatrick (1975) problem.

The first, partial, test of stability near T_c was done by Thouless *et al* (1980). Next a complete stability analysis, but still confined to the neighbourhood of T_c, appeared (De Dominicis and Kondor 1983), which was also extended to the case of the Sompolinsky (1981) solution, displaying an identical fluctuation spectrum, and including a finite external field (Kondor and De Dominicis 1983). An independent calculation by Goltsev (1983) for the Parisi case followed. This last paper contains partial results valid for all $T < T_c$, which are based on a particular type of eigenfunction and do not constitute a complete analysis, but in the vicinity of T_c it also gives a complete stability test, which can therefore be compared with ours. Our comments here concern this second half of Goltsev's paper.

First of all we have to point out a minor difference in convention. Both Goltsev and ourselves start with the same truncated free energy functional, but he gives the eigenvalues of the matrix of second derivatives, whereas we give the eigenvalues of twice the same matrix, hence all our eigenvalues should be twice his. (Our convention is chosen to let the eigenvalues become the free squared masses of a future field theory of spin glasses.)

Both he and we find two continuous bands of eigenvalues: the band of small, $O(\tau^2)$, eigenvalues, and the band of large, $O(\tau)$, eigenvalues ($\tau = (T_c - T)/T_c$). Our small eigenvalues span the range $(0, 2\tau^2)$, while his small eigenvalues, as given by his equation (3.18), span the same range, whereas in view of the factor 2 difference, they should span $(0, \tau^2)$. In fact, there is an obvious misprint in his equation (3.18): combining his equations (3.15), (3.17*a*), (3.17*b*) one immediately sees that in place of the factor $\frac{1}{2}$ one should have $\frac{1}{4}$ in his equation (3.18), which then brings the upper edge of the small band in the two calculations to agreement.

As for the large eigenvalues, Goltsev's equation (3.19) implies

$$\tau + \tau^2 + \tfrac{8}{3}\tau^3 + O(\tau^4) \leqslant \lambda \leqslant \tau + \tau^2 + \tfrac{10}{3}\tau^3 + O(\tau^4). \tag{1}$$

We assert that this band is too narrow and somewhat too high. Indeed, consider the large eigenvalue in our 'first family' as given by equation (3) in De Dominicis and Kondor (1983). A trivial expansion to $O(\tau^2)$ yields $\lambda = 2\tau - \frac{2}{3}\tau^2 + \cdots$, which, taking proper account of the overall factor 2 difference, is seen to fall below the lower edge of Goltsev's band. Now this particular eigenvalue (like the whole of our first family) was already implicit in Thouless *et al* (1980), see also de Almeida's thesis (1980), and it was obtained by a method totally different from ours, which leaves little doubt about its being correct.

The rest of our large eigenvalues can be obtained with equal ease from equations (7), (8), (11) in De Dominicis and Kondor (1983) and yield the band

$$2\tau - \tfrac{2}{3}\tau^2 + O(\tau^3) \leqslant \lambda \leqslant 2\tau + \tfrac{2}{3}\tau^2 + O(\tau^3) \tag{2}$$

which is wider than Goltsev's and does not overlap with it. Furthermore, a study of the same equations shows that there are no other large eigenvalues in the spectrum, all the other solutions fall below the upper edge of the small band.

The root of this discrepancy can be traced back to Goltsev's equation (3.10). This formula, as it stands, cannot be right. It says that the determinant of the Hessian contains factors like $1 - D_j(\lambda)$ where $D_j(\lambda)$ is some complicated expression defined in (3.12*a*), (3.12*b*), but then the author determines the roots of the determinant from $D_j(\lambda) = 0$. Evidently, $1 - D_j$ is a misprint then, and has to be replaced probably by D_j everywhere in (3.10). So doing one can at least reproduce the known spectrum for the replica symmetric case ($q_j \equiv q$), which is impossible otherwise. (Equation (3.10) in its original form does not even give the right number of roots).

The question is now whether (3.10), corrected as proposed, can also produce the right spectrum if there is symmetry breaking. The simplest imaginable test is to consider a 4×4 order parameter matrix ($n = 4$) divided into 2×2 blocks ($m_1 = 2$, $K = 1$) with matrix elements q_1 in the off-diagonal blocks, q_0 in the diagonal ones. In fact Goltsev's equation (3.7) should obviously contain $\tilde{q}_j (\equiv q_j - q_{j+1})$ instead of q_j, a misprint that crept in at the printer's stage. The Hessian corresponding to this case is

$$\begin{pmatrix}
\lambda+\tau+q_0^2 & q_1/2 & q_1/2 & q_1/2 & q_1/2 & 0 \\
q_1/2 & \lambda+\tau+q_1^2 & q_0/2 & q_0/2 & 0 & q_1/2 \\
q_1/2 & q_0/2 & \lambda+\tau+q_1^2 & 0 & q_0/2 & q_1/2 \\
q_1/2 & q_0/2 & 0 & \lambda+\tau+q_1^2 & q_0/2 & q_1/2 \\
q_1/2 & 0 & q_0/2 & q_0/2 & \lambda+\tau+q_1^2 & q_1/2 \\
0 & q_1/2 & q_1/2 & q_1/2 & q_1/2 & \lambda+\tau+q_0^2
\end{pmatrix}. \tag{3}$$

The notation conforms to Goltsev's convention here.

This matrix is trivial to diagonalise directly, i.e. without invoking any special methods like his or ours. The spectrum is

$$\lambda = \begin{cases} q_0 - \tau - q_1^2 & \\ -\tau - q_0^2 & \\ -\tau - q_1^2, & \text{doubly degenerate,} \end{cases} \tag{4}$$

plus the two roots of

$$(\lambda + \tau + q_0^2)(\lambda + \tau + q_0 + q_1^2) = 2q_1^2.$$

On the other hand, from Goltsev's equations (3.10 corrected), (3.11), (3.12*a*) one obtains

$$\lambda = \begin{cases} q_0 - \tau - q_0^2 \\ q_0 - \tau - q_1^2 \\ \quad - \tau - q_0^2 \\ \quad - \tau - q_1^2 \end{cases} \tag{5}$$

plus the roots of

$$(\lambda + \tau + q_0^2)(\lambda + \tau + q_0 + q_1^2) = 2q_0^2.$$

The difference between (4) and (5) shows that Goltsev's equation (3.10), which is meant to contain the full spectrum of fluctuations around T_c, is wrong. We note that the eigenvectors as described in our paper lead to the correct spectrum, equation (4), in the test case above.

To conclude we point out that the discrepancies between Goltsev's spectrum and ours go far beyond some small quantitative differences. As a matter of fact, the agreement between his small band and ours is deceptive. Though the roots given by his equation (3.15) do belong to the spectrum and span, in the continuous case, the right range, they do not exhaust all the small eigenvalues. As described in our paper, the third family cannot be parametrised by a single continuous variable like in his equation (3.15), but takes three of them. (A special case of this, a two-parameter subfamily, has also been found by Sompolinsky and Zippelius (1983) in a totally different approach.) From a field theoretic point of view this means that the free propagator of the theory will have a spectral function depending on three continuous labels, certainly a case of unprecedented complexity. The infrared behaviour of the theory may depend crucially on this feature.

References

de Almeida J R L 1980 *PhD Thesis, Birmingham*, unpublished
De Dominicis C and Kondor I 1983 *Phys. Rev.* B Rapid Comm. **27** 606
Goltsev A V 1983 *J. Phys. A: Math. Gen.* **16** 1337
Kondor I and De Dominicis C 1983 *J. Phys. A: Math. Gen.* **16** L73
Parisi G 1979 *Phys. Rev. Lett.* **43** 1754
—— 1980 *J. Phys. A: Math. Gen.* **13** L115, 1101, 1887
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
Sompolinsky H 1981 *Phys. Rev. Lett.* **47** 935
Sompolinsky H and Zippelius A 1983 *Phys. Rev. Lett.* **50** 1297
Thouless D J, de Almeida J R L and Kosterlitz J M 1980 *J. Phys. C: Solid State Phys.* **13** 3272